

FURTHER RESULTS ON GENERALIZED INVERSES OF TENSORS VIA THE EINSTEIN PRODUCT

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Abstract

The notion of the Moore-Penrose inverse of tensors with the Einstein product was introduced, very recently. In this paper, we further elaborate this theory by producing a few characterizations of different generalized inverses of tensors. A new method to compute the Moore-Penrose inverse of tensors is proposed. Reverse order laws for several generalized inverses of tensors are also presented. In addition to these, we discuss general solutions of multilinear systems of tensors using such theory.

Keywords: Moore-Penrose inverse; Tensor; Matrix; Einstein product.

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1. Introduction

1.1. Background and motivation

The Moore-Penrose inverse of an arbitrary matrix (including singular and rectangular) has many applications in statistics, prediction theory, control system analysis, curve fitting, numerical analysis and solution of linear integral equations. But increasing ability of data collection systems to store huge volumes of multidimensional data and the recognition of potential modeling accuracy, matrix representation of data analysis is not enough to represent all the information content of the multiway data in different fields, including computer image and human motion recognition [2], signal processing [16, 17], and many other areas using multiway data analysis [4, 15]. There has been a recent surge in the research and utilization of tensors (see the articles [7, 11, 13]) which are high-order generalization of matrices.

Tensor models are employed in numerous disciplines addressing the problem of finding the multilinear structure in multiway data sets. Multilinear systems model many phenomena in engineering and science. For example, in continuum physics and engineering, isotropic and anisotropic elasticity are modelled [8] as multilinear systems. Further, the reverse order laws for generalized inverses of matrices [1] play an important role in theoretic research and numerical computations in ill-posed problems, optimization problems, and statistics problems. In addition, the reverse order laws for generalized inverses are also applied to the generalized least squares problem and the weighted perturbation theory. It will be more applicable if we investigate reverse order laws for generalized inverses of tensors, and hence reverse order of generalized inverses of tensors will open different paths of the above areas. It is thus of interest to study theory of generalized inverses of a tensor via the Einstein product and its applications. In this direction, the Moore-Penrose inverse of a tensor was introduced in [18], very recently, and then the authors ([18]) used the same notion to solve multilinear systems of tensors.

In this paper, we further study on generalized inverses of tensors. This study can lead to the enhancement of the computation of generalized inverses of tensors along with solutions of multilinear structure in multidimensional systems. In this regard, we introduce different generalized inverses of tensors and then provide different characterizations of generalized inverses of tensors.

We also present reverse order laws for several generalized inverses of tensors. Besides these, a new method for computing the Moore-Penrose inverse of a tensor is obtained. Further, applications of these notions to multilinear systems are studied, where we present a different approach to prove Theorem 4.1, [18].

1.2. Outline

We organize the paper as follows. In the next subsection, we introduce some notations and definitions which are helpful in proving the main results in Section 2 and Section 3. In Section 2, we prove several results concerning different generalized inverses. In particular, we propose a method for computing the Moore-Penrose inverse of a tensor. A few applications of these results to multilinear systems of tensors are discussed in Section 3.

1.3. Notations and definitions

Multiway arrays, often referred to as tensors, are higher-order generalizations of vectors and matrices. For a positive integer N , let $[N] = \{1, \dots, N\}$. An order N tensor $\mathcal{A} = (a_{i_1 \dots i_N})_{1 \leq i_j \leq I_j} (j = 1, \dots, N)$ is a multidimensional array with $I_1 I_2 \dots I_N$ entries. Let $\mathbb{C}^{I_1 \times \dots \times I_N}$ and $\mathbb{R}^{I_1 \times \dots \times I_N}$ be the sets of the order N dimension $I_1 \times \dots \times I_N$ tensors over the complex field \mathbb{C} and the real field \mathbb{R} , respectively. For instance, $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N}$ is a multiway array with N -th order and I_1, I_2, \dots, I_N dimension in the first, second, \dots , N th way, respectively. Each entry of \mathcal{A} is denoted by $a_{i_1 \dots i_N}$. For $N = 3$, $\mathcal{A} \in \mathbb{C}^{I_1 \times I_2 \times I_3}$ is a third order tensor, and $a_{i_1 i_2 i_3}$ denotes the entry of that tensor. Many results on tensors which have attracted great interest can be found in the articles [7, 13].

For a tensor $\mathcal{A} = (a_{i_1 \dots i_N j_1 \dots j_M}) \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_M}$, let $\mathcal{B} = (b_{i_1 \dots i_M j_1 \dots j_N}) \in \mathbb{C}^{J_1 \times \dots \times J_M \times I_1 \times \dots \times I_N}$ be the conjugate transpose of \mathcal{A} , where $b_{i_1 \dots i_M j_1 \dots j_N} = \overline{a_{j_1 \dots j_M i_1 \dots i_N}}$. The tensor \mathcal{B} is denoted by \mathcal{A}^* . When $b_{i_1 \dots i_M j_1 \dots j_N} = a_{j_1 \dots j_M i_1 \dots i_N}$, \mathcal{B} is the *transpose* of \mathcal{A} , and is denoted by \mathcal{A}^T . A tensor $\mathcal{D} = (d_{i_1 \dots i_N i_1 \dots i_N}) \in \mathbb{C}^{I_1 \times \dots \times I_N \times I_1 \times \dots \times I_N}$ is called a *diagonal tensor* if all its entries are zero except for $d_{i_1 \dots i_N i_1 \dots i_N}$. In case of all the diagonal entries $d_{i_1 \dots i_N i_1 \dots i_N} = 1$, we call \mathcal{D} as a *unit tensor*, and is denoted by \mathcal{I} . Similarly, \mathcal{O} denotes the zero tensor in case of all the entries are zero. A few more notations and definitions are introduced below for defining generalized inverses of a tensor.

For $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times K_1 \times \dots \times K_N}$ and $\mathcal{B} \in \mathbb{C}^{K_1 \times \dots \times K_N \times J_1 \times \dots \times J_M}$, the Einstein product [5] of tensors \mathcal{A} and \mathcal{B} is defined by the operation $*_N$ via

$$(\mathcal{A} *_N \mathcal{B})_{i_1 \dots i_N j_1 \dots j_M} = \sum_{k_1 \dots k_N} a_{i_1 \dots i_N k_1 \dots k_N} b_{k_1 \dots k_N j_1 \dots j_M}. \quad (1.1)$$

Note that $\mathcal{A} *_N \mathcal{B} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_M}$. The associative law of this tensor product holds. In the above formula, when $\mathcal{B} \in \mathbb{C}^{K_1 \times \dots \times K_N}$, then

$$(\mathcal{A} *_N \mathcal{B})_{i_1 \dots i_N} = \sum_{k_1 \dots k_N} a_{i_1 \dots i_N k_1 \dots k_N} b_{k_1 \dots k_N}, \quad (1.2)$$

where $\mathcal{A} *_N \mathcal{B} \in \mathbb{C}^{I_1 \times \dots \times I_N}$. On the other hand, when $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N}$ and \mathcal{B} is a vector $b = (b_i) \in \mathbb{C}^{I_N}$, the product is defined by the operation \times_N via

$$(\mathcal{A} \times_N b)_{i_1 \dots i_{N-1}} = \sum_{i_N} a_{i_1 \dots i_N} b_{i_N}, \quad (1.3)$$

where the tensor $\mathcal{A} \times_N b \in \mathbb{C}^{I_1 \times \dots \times I_{N-1}}$. We next collect some definitions and results which are essential to prove our main results.

Definition 1.1. [18] For a tensor $\mathcal{A} = (a_{i_1 \dots i_N j_1 \dots j_M}) \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_M}$, $\mathcal{A}_{(i_1 \dots i_N | :)} = (a_{i_1 \dots i_N : \dots :}) \in \mathbb{C}^{J_1 \times \dots \times J_M}$ is a subblock of \mathcal{A} . $\text{Vec}(\mathcal{A})$ is obtained by lining up all the subtensors in a column, and t -th subblock of $\text{Vec}(\mathcal{A})$ is $\mathcal{A}_{(i_1 \dots i_N | :)}^{(t)}$, where $t = i_N + \sum_{K=1}^{N-1} \left[(i_K - 1) \prod_{L=K+1}^N I_L \right]$.

Definition 1.2. [18] The Kronecker product of $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$ and $\mathcal{B} \in \mathbb{C}^{K_1 \times \dots \times K_M \times L_1 \times \dots \times L_M}$, denoted by $\mathcal{A} \otimes \mathcal{B} = (a_{i_1 \dots i_N j_1 \dots j_N} \mathcal{B})$, is a ‘Kr-block tensor’ whose (t_1, t_2) subblock is $(a_{i_1 \dots i_N j_1 \dots j_N} \mathcal{B})^{(t_1, t_2)}$ obtained via multiplying all the entries of \mathcal{B} by a constant $a_{i_1 \dots i_N j_1 \dots j_N}$, where

$$t_1 = i_N + \sum_{K=1}^{N-1} \left[(i_K - 1) \prod_{L=K+1}^N I_L \right] \text{ and } t_2 = j_N + \sum_{K=1}^{N-1} \left[(j_K - 1) \prod_{L=K+1}^N J_L \right].$$

Remark 1. From the above definition, it can be immediately seen that the Kronecker product is not commutative, i.e., $\mathcal{A} \otimes \mathcal{B} \neq \mathcal{B} \otimes \mathcal{A}$.

The next result can be verified easily.

Proposition 1.3. Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$, $\mathcal{B} \in \mathbb{C}^{K_1 \times \dots \times K_M \times L_1 \times \dots \times L_M}$, $\mathcal{C} \in \mathbb{C}^{K_1 \times \dots \times K_M \times L_1 \times \dots \times L_M}$ and $\mathcal{D} \in \mathbb{C}^{J_1 \times \dots \times J_N \times L_1 \times \dots \times L_M}$. Then

- (a) $(\mathcal{A} \otimes \mathcal{B})^* = \mathcal{A}^* \otimes \mathcal{B}^*$;
- (b) $\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) = (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}$;
- (c) $\mathcal{A} \otimes (\mathcal{B} + \mathcal{C}) = \mathcal{A} \otimes \mathcal{B} + \mathcal{A} \otimes \mathcal{C}$ and $(\mathcal{B} + \mathcal{C}) \otimes \mathcal{A} = \mathcal{B} \otimes \mathcal{A} + \mathcal{C} \otimes \mathcal{A}$;
- (d) $(\mathcal{A} \otimes \mathcal{B}) *_M \text{Vec}(\mathcal{D}) = \text{Vec}(\mathcal{A} *_N \mathcal{D} *_M \mathcal{B}^T)$.

We have another result presented below on Kronecker product of tensors.

Lemma 1.4. (Proposition 2.3, [18])

Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$, $\mathcal{B} \in \mathbb{C}^{K_1 \times \dots \times K_M \times L_1 \times \dots \times L_M}$, $\mathcal{C} \in \mathbb{C}^{J_1 \times \dots \times J_N \times I_1 \times \dots \times I_N}$ and $\mathcal{D} \in \mathbb{C}^{L_1 \times \dots \times L_M \times K_1 \times \dots \times K_M}$. Then

$$(\mathcal{A} \otimes \mathcal{B}) *_M (\mathcal{C} \otimes \mathcal{D}) = (\mathcal{A} *_N \mathcal{C}) \otimes (\mathcal{B} *_M \mathcal{D}).$$

2. Generalized inverses of tensors

This section begins with the definitions of various generalized inverses of a tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$ via the Einstein product. We first recall the definition of the Moore-Penrose inverse of a tensor which was introduced in [18], very recently.

Definition 2.1. (Definition 2.2, [18]) Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$. The tensor $\mathcal{X} \in \mathbb{C}^{J_1 \times \dots \times J_N \times I_1 \times \dots \times I_N}$ satisfying the following four tensor equations:

- (1) $\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{A}$;
- (2) $\mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}$;
- (3) $(\mathcal{A} *_N \mathcal{X})^* = \mathcal{A} *_N \mathcal{X}$;
- (4) $(\mathcal{X} *_N \mathcal{A})^* = \mathcal{X} *_N \mathcal{A}$,

is called the **Moore-Penrose inverse** of \mathcal{A} , and is denoted by \mathcal{A}^\dagger .

Let λ be a nonempty subset of $\{1, 2, 3, 4\}$. Then a tensor $\mathcal{X} \in \mathbb{C}^{J_1 \times \dots \times J_N \times I_1 \times \dots \times I_N}$ is called a $\{\lambda\}$ -inverse of \mathcal{A} if \mathcal{X} satisfies equation (i) for each $i \in \lambda$. We denote such an inverse by $\mathcal{A}^{(\lambda)}$ while $\mathcal{A}\{\lambda\}$ stands for the class of all $\{\lambda\}$ -inverses of \mathcal{A} . For $\lambda = \{1\}$ and $\lambda = \{1, 2\}$, \mathcal{X} is called as a *generalized*

inverse and a *reflexive generalized inverse* of \mathcal{A} , respectively. If $\lambda = \{1, 3\}$, then we have $\{1, 3\}$ -*inverse* of \mathcal{A} , and for $\lambda = \{1, 4\}$, we get $\{1, 4\}$ -*inverse* of \mathcal{A} .

Brazell et al. [2] introduced the notion of ordinary tensor inverse, and is as follows. A tensor $\mathcal{X} \in \mathbb{C}^{I_1 \times \dots \times I_N \times I_1 \times \dots \times I_N}$ is called the *inverse* of $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times I_1 \times \dots \times I_N}$ if it satisfies $\mathcal{A} *_N \mathcal{X} = \mathcal{X} *_N \mathcal{A} = \mathcal{I}$. It is denoted by \mathcal{A}^{-1} . A tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times I_1 \times \dots \times I_N}$ is *hermitian* if $\mathcal{A} = \mathcal{A}^*$. Further, a tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times I_1 \times \dots \times I_N}$ is *unitary* if $\mathcal{A} *_N \mathcal{A}^* = \mathcal{A}^* *_N \mathcal{A} = \mathcal{I}$, and *idempotent* if $\mathcal{A} *_N \mathcal{A} = \mathcal{A}$. In case of tensors of real entries, hermitian and unitary tensors are called *symmetric* and *orthogonal* tensors. (See Definition 3.15 and Definition 3.16 of [2], respectively.)

In case of an invertible tensor \mathcal{A} , $\mathcal{A}^\dagger = \mathcal{A}^{(\lambda)} = \mathcal{A}^{-1}$. Next, we collect some known properties of \mathcal{A}^\dagger (see Proposition 3.3, [18]) which will be frequently used in this paper: $(\mathcal{A}^\dagger)^\dagger = \mathcal{A}$ and $(\mathcal{A}^*)^\dagger = (\mathcal{A}^\dagger)^*$. We now move to prove a result which was also proposed in Proposition 3.3 (4), [18], and to do this we need the following lemma which we call as singular value decomposition (SVD) of a tensor proved in Theorem 3.17, [2] for a real tensor. The authors of [18] stated the same result for a complex tensor, and is recalled next.

Lemma 2.2. (Lemma 3.1, [18]) A tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$ can be decomposed as

$$\mathcal{A} = \mathcal{U} *_N \mathcal{B} *_N \mathcal{V}^*,$$

where $\mathcal{U} \in \mathbb{C}^{I_1 \times \dots \times I_N \times I_1 \times \dots \times I_N}$ and $\mathcal{V} \in \mathbb{C}^{J_1 \times \dots \times J_N \times J_1 \times \dots \times J_N}$ are unitary tensors, $\mathcal{B} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$ is a tensor such that $(\mathcal{B})_{i_1 \dots i_N j_1 \dots j_N} = 0$, if $(i_1, \dots, i_N) \neq (j_1, \dots, j_N)$.

Existence and uniqueness of \mathcal{A}^\dagger is shown in Theorem 3.2, [18]. The authors of [18] also showed that $\mathcal{A}^\dagger = \mathcal{V} *_N \mathcal{B}^\dagger *_N \mathcal{U}^*$ in the proof of Theorem 3.2, [18]. Using this, we will prove the condition (a) in the next result.

Lemma 2.3. Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$. Then

- (a) $(\mathcal{A}^* *_N \mathcal{A})^\dagger = \mathcal{A}^\dagger *_N (\mathcal{A}^*)^\dagger$.
- (b) $\mathcal{A}^\dagger = (\mathcal{A}^* *_N \mathcal{A})^\dagger *_N \mathcal{A}^*$.

Proof. (a) As an application of SVD of \mathcal{A} , we have $\mathcal{A}^\dagger = \mathcal{V} *_N \mathcal{B}^\dagger *_N \mathcal{U}^*$.

Hence

$$\begin{aligned}
(\mathcal{A}^* *_N \mathcal{A})^\dagger &= (\mathcal{V} *_N \mathcal{B}^* *_N \mathcal{U}^* *_N \mathcal{U} *_N \mathcal{B} *_N \mathcal{V}^*)^\dagger \\
&= (\mathcal{V} *_N (\mathcal{B}^* *_N \mathcal{B})^\dagger *_N \mathcal{V}^* \\
&= \mathcal{V} *_N \mathcal{B}^\dagger *_N (\mathcal{B}^*)^\dagger *_N \mathcal{V}^* \\
&= \mathcal{V} *_N \mathcal{B}^\dagger *_N \mathcal{U}^* *_N \mathcal{U} *_N (\mathcal{B}^*)^\dagger *_N \mathcal{V}^* \\
&= \mathcal{A}^\dagger *_N \mathcal{A}^{*\dagger}.
\end{aligned}$$

(b) Let $\mathcal{C} = \mathcal{A}^* *_N \mathcal{A}$, $\mathcal{B} = (\mathcal{A}^* *_N \mathcal{A})^\dagger *_N \mathcal{A}^* = \mathcal{C}^\dagger *_N \mathcal{A}^*$ and $\mathcal{X} = \mathcal{A}$. Then $\mathcal{B} *_N \mathcal{X} = \mathcal{C}^\dagger *_N \mathcal{C}$. So $\mathcal{B} *_N \mathcal{X}$ is hermitian. Then it follows that $\mathcal{B} *_N \mathcal{X} *_N \mathcal{B} = \mathcal{B}$ by using condition (2) of Definition 2.1 for tensor \mathcal{C} . Similarly, $\mathcal{X} *_N \mathcal{B} = \mathcal{A} *_N (\mathcal{A}^* *_N \mathcal{A})^\dagger *_N \mathcal{A}^*$. So \mathcal{X} satisfies condition (4) of Definition 2.1. The remained condition (2) is shown hereunder.

$$\begin{aligned}
\mathcal{X} *_N \mathcal{B} *_N \mathcal{X} &= \mathcal{A} *_N (\mathcal{A}^* *_N \mathcal{A})^\dagger *_N \mathcal{A}^* *_N \mathcal{A} \\
&= \mathcal{A} *_N \mathcal{A}^\dagger *_N (\mathcal{A}^*)^\dagger *_N \mathcal{A}^* *_N \mathcal{A} \\
&= \mathcal{A} *_N \mathcal{A}^\dagger *_N (\mathcal{A} *_N \mathcal{A}^\dagger)^* *_N \mathcal{A} \\
&= \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A} = \mathcal{X}.
\end{aligned}$$

□

Remark 2. Nevertheless, Lemma 2.3 (a) is not true if we replace \mathcal{A}^* by any other tensor \mathcal{B} , i.e., $(\mathcal{B} *_N \mathcal{A})^\dagger \neq \mathcal{B}^\dagger *_N \mathcal{A}^\dagger$, where \mathcal{A} and $\mathcal{B} \in \mathbb{C}^{I_1 \times \dots \times I_N \times I_1 \times \dots \times I_N}$.

Example 2.4. Let $\mathcal{A} = (a_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$ and $\mathcal{B} = (b_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$ be two tensors such that

$$a_{ij11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, a_{ij21} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, a_{ij12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, a_{ij22} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix},$$

and

$$b_{ij11} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, b_{ij21} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b_{ij12} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, b_{ij22} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

respectively. Then $\mathcal{A}^\dagger = (x_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$ and $\mathcal{B}^\dagger = (y_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, where

$$x_{ij11} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, x_{ij21} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, x_{ij12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, x_{ij22} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$y_{ij11} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, y_{ij21} = \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}, y_{ij12} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, y_{ij22} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

respectively. So $\mathcal{B}^\dagger *_N \mathcal{A}^\dagger = (d_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, where

$$d_{ij11} = \begin{pmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}, d_{ij21} = \begin{pmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}, d_{ij12} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, d_{ij22} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

and $(\mathcal{A} *_N \mathcal{B})^\dagger = (c_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, where

$$c_{ij11} = \begin{pmatrix} 0 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}, c_{ij21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, c_{ij12} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, c_{ij22} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Hence

$$(\mathcal{A} *_N \mathcal{B})^\dagger \neq \mathcal{B}^\dagger *_N \mathcal{A}^\dagger.$$

Some sufficient conditions are obtained below for the equality case.

Proposition 2.5. For $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$ and $\mathcal{B} \in \mathbb{C}^{J_1 \times \dots \times J_N \times I_1 \times \dots \times I_N}$, $(\mathcal{A} *_N \mathcal{B})^\dagger = \mathcal{B}^\dagger *_N \mathcal{A}^\dagger$, if one of the following conditions holds.

- (a) $\mathcal{B} = \mathcal{A}^*$.
- (b) $\mathcal{B} = \mathcal{A}^\dagger$.
- (c) $\mathcal{A}^* *_N \mathcal{A} = \mathcal{I}$.
- (d) $\mathcal{B} *_N \mathcal{B}^* = \mathcal{I}$.

Using the method as in the proof of Lemma 2.3, one can prove the next lemma.

Lemma 2.6. Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$. Then the following are true.

- (a) $\mathcal{A}^\dagger = \mathcal{A}^* *_N (\mathcal{A} *_N \mathcal{A}^*)^\dagger$.
- (b) $\mathcal{A}^* = \mathcal{A}^* *_N \mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{A}^*$.
- (c) $\mathcal{A} = \mathcal{A} *_N \mathcal{A}^* *_N (\mathcal{A}^*)^\dagger = (\mathcal{A}^*)^\dagger *_N \mathcal{A}^* *_N \mathcal{A}$.
- (d) $(\mathcal{U} *_N \mathcal{A} *_N \mathcal{V})^\dagger = \mathcal{V}^* *_N \mathcal{A}^\dagger *_N \mathcal{U}^*$, where \mathcal{U} and \mathcal{V} are unitary tensors.
- (e) $\mathcal{A}^\dagger *_N \mathcal{A} = \mathcal{A}^* *_N (\mathcal{A}^*)^\dagger$ and $\mathcal{A} *_N \mathcal{A}^\dagger = (\mathcal{A}^*)^\dagger *_N \mathcal{A}^*$.

Using Lemma 2.6 (b) and (c), one can prove the result obtained below.

Corollary 2.7. *Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ and $\mathcal{A} *_N \mathcal{A}^\dagger = \mathcal{A}^\dagger *_N \mathcal{A}$. Then*

- (a) *there exists a $\mathcal{X} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ such that $\mathcal{A} *_N \mathcal{X} = \mathcal{A}^*$, and*
- (b) *there exists a $\mathcal{Y} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ such that $\mathcal{A}^* *_N \mathcal{Y} = \mathcal{A}$.*

Similarly, the following lemma can be proved.

Lemma 2.8. *Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$. Then the following are true.*

- (a) *If \mathcal{A} is hermitian and idempotent, then $\mathcal{A}^\dagger = \mathcal{A}$.*
- (b) *$\mathcal{A}^\dagger = \mathcal{A}^*$ if and only if $\mathcal{A}^* *_N \mathcal{A}$ is idempotent.*
- (c) *$\mathcal{A} *_N \mathcal{A}^\dagger$, $\mathcal{A}^\dagger *_N \mathcal{A}$, $\mathcal{I} - \mathcal{A} *_N \mathcal{A}^\dagger$ and $\mathcal{I} - \mathcal{A}^\dagger *_N \mathcal{A}$ are all idempotent.*

Analogous results to all the above discussed results for matrices can be found in Chapter 1, [1]. We now proceed to discuss a few properties of $\{1\}$ -inverse of a tensor, below.

Theorem 2.9. *The following three conditions are equivalent:*

- (i) $(\mathcal{A}^{(1)})^* \{1\} = (\mathcal{A}^*)^{(1)} \{1\}$.
- (ii) $\mathcal{A} *_N (\mathcal{A}^* *_N \mathcal{A})^{(1)} *_N \mathcal{A}^* *_N \mathcal{A} = \mathcal{A}$.
- (iii) $(\mathcal{A} *_N (\mathcal{A}^* *_N \mathcal{A})^{(1)} *_N \mathcal{A}^*)^* = \mathcal{A} *_N (\mathcal{A}^* *_N \mathcal{A})^{(1)} *_N \mathcal{A}^*$.

Proof. (i) The condition $\mathcal{A} *_N \mathcal{A}^{(1)} *_N \mathcal{A} = \mathcal{A}$ implies $\mathcal{A}^* *_N (\mathcal{A}^{(1)})^* *_N \mathcal{A}^* = \mathcal{A}^*$. Hence $(\mathcal{A} \{1\})^* \subseteq \mathcal{A}^* \{1\}$. The other implication follows from $(\mathcal{A}^* \{1\})^* \subseteq \mathcal{A} \{1\}$ because of $\mathcal{A}^* *_N (\mathcal{A}^*)^{(1)} *_N \mathcal{A}^* = \mathcal{A}^*$.

(ii) Let $\mathcal{R} = (\mathcal{I} - \mathcal{A} *_N (\mathcal{A}^* *_N \mathcal{A})^{(1)} *_N \mathcal{A}^*) *_N \mathcal{A} = \mathcal{A} *_N (\mathcal{I} - (\mathcal{A}^* *_N \mathcal{A})^{(1)} *_N \mathcal{A}^* *_N \mathcal{A})$. Then

$$\begin{aligned} & \mathcal{R}^* *_N \mathcal{R} \\ &= (\mathcal{I} - (\mathcal{A}^* *_N \mathcal{A})^{(1)} *_N \mathcal{A}^* *_N \mathcal{A})^* *_N (\mathcal{A}^* *_N \mathcal{A} - \mathcal{A}^* *_N \mathcal{A} *_N (\mathcal{A}^* *_N \mathcal{A})^{(1)} *_N \mathcal{A}^* *_N \mathcal{A}) \\ &= \mathcal{O}. \end{aligned}$$

We thus have $\mathcal{R} = \mathcal{O}$ which implies the desired result.

(iii) Let \mathcal{R} be a $\{1\}$ -inverse of $\mathcal{A}^* *_N \mathcal{A}$. Then, by (i), we have \mathcal{R}^* as a $\{1\}$ -inverse of $\mathcal{A}^* *_N \mathcal{A}$. Again, setting $\mathcal{S} = (\mathcal{R} + \mathcal{R}^*)/2$, we get \mathcal{S} as $\{1\}$ -inverse of $\mathcal{A}^* *_N \mathcal{A}$, and also \mathcal{S} is hermitian. Let $\mathcal{G} = \mathcal{A} *_N \mathcal{S} *_N \mathcal{A}^* -$

$\mathcal{A} *_N (\mathcal{A}^* *_N \mathcal{A})^{(1)} *_N \mathcal{A}^*$. Then

$$\begin{aligned} & \mathcal{G}^* *_N \mathcal{G} \\ = & (\mathcal{A} *_N \mathcal{S} - \mathcal{A} *_N (\mathcal{A}^* *_N \mathcal{A})^{(1)})^* *_N (\mathcal{A}^* *_N \mathcal{A} *_N \mathcal{S} *_N \mathcal{A}^* - \mathcal{A}^* *_N \mathcal{A} *_N (\mathcal{A}^* *_N \mathcal{A})^{(1)} *_N \mathcal{A}^*) \\ = & \mathcal{O}. \end{aligned}$$

Hence $\mathcal{G} = \mathcal{O}$ which leads to (iii). □

The matrix version of the above result is obtained below for $A \in \mathbb{C}^{m \times n}$.

Corollary 2.10. *The following three conditions are equivalent:*

- (i) $A^{(1)*} \{1\} = (A^*)^{(1)} \{1\}$.
- (ii) $A(A^*A)^{(1)}A^*A = A$.
- (iii) $(A(A^*A)^{(1)}A^*)^* = A(A^*A)^{(1)}A^*$.

Next result collects sufficient conditions for $(\mathcal{A} + \mathcal{B})^{(1)}$ to be a $\{1\}$ -inverse of \mathcal{B} .

Theorem 2.11. *Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$. If $\mathcal{A}^{(1)} *_N \mathcal{B} = \mathcal{B}^{(1)} *_N \mathcal{A} = \mathcal{O}$, then $(\mathcal{A} + \mathcal{B})^{(1)}$ is a $\{1\}$ -inverse of \mathcal{B} .*

Proof. The condition $\mathcal{A}^{(1)} *_N \mathcal{B} = \mathcal{O}$ yields $\mathcal{B} = \mathcal{A} - \mathcal{A} *_N \mathcal{A}^{(1)} *_N \mathcal{A} + \mathcal{B} - \mathcal{A} *_N \mathcal{A}^{(1)} *_N \mathcal{B} = (\mathcal{I} - \mathcal{A} *_N \mathcal{A}^{(1)}) *_N (\mathcal{A} + \mathcal{B})$. Using the other condition, one can have $\mathcal{B} = (\mathcal{A} + \mathcal{B}) *_N (\mathcal{I} - \mathcal{A}^{(1)} *_N \mathcal{A})$. Then

$$\begin{aligned} & \mathcal{B} *_N (\mathcal{A} + \mathcal{B})^{(1)} *_N \mathcal{B} \\ = & (\mathcal{I} - \mathcal{A} *_N \mathcal{A}^{(1)}) *_N (\mathcal{A} + \mathcal{B}) *_N (\mathcal{A} + \mathcal{B})^{(1)} *_N (\mathcal{A} + \mathcal{B}) *_N (\mathcal{I} - \mathcal{A}^{(1)} *_N \mathcal{A}) \\ = & (\mathcal{I} - \mathcal{A} *_N \mathcal{A}^{(1)}) *_N (\mathcal{A} + \mathcal{B}) *_N (\mathcal{I} - \mathcal{A}^{(1)} *_N \mathcal{A}) \\ = & (\mathcal{I} - \mathcal{A} *_N \mathcal{A}^{(1)}) *_N \mathcal{B} \\ = & \mathcal{B} - \mathcal{A} *_N \mathcal{A}^{(1)} *_N \mathcal{B} = \mathcal{B}. \end{aligned}$$

□

The following one is obtained as a corollary for matrices.

Corollary 2.12. *Let $A, B \in \mathbb{C}^{m \times n}$. If $A^{(1)}B = B^{(1)}A = O$, then $(A + B)^{(1)}$ is a $\{1\}$ -inverse of B .*

Observe that $\mathcal{B} = \mathcal{A} *_N \mathcal{A}^{(1)} *_N \mathcal{B}$ implies $\mathcal{B} = \mathcal{A} *_N \mathcal{H}$ where $\mathcal{H} = \mathcal{A}^{(1)} *_N \mathcal{B}$. Conversely, if $\mathcal{B} = \mathcal{A} *_N \mathcal{H}$, then premultiplying $\mathcal{A} *_N \mathcal{A}^{(1)}$, we have

$$\mathcal{A} *_N \mathcal{A}^{(1)} *_N \mathcal{B} = \mathcal{A} *_N \mathcal{A}^{(1)} *_N \mathcal{A} *_N \mathcal{H} = \mathcal{A} *_N \mathcal{H} = \mathcal{B}.$$

Hence, in this case, $\{1\}$ -inverse of a tensor behaves like ordinary inverse of a tensor which is stated in the next result.

Theorem 2.13. *Let $\mathcal{A}, \mathcal{B} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$. Then $\mathcal{B} = \mathcal{A} *_N \mathcal{A}^{(1)} *_N \mathcal{B}$ if and only if $\mathcal{B} = \mathcal{A} *_N \mathcal{H}$ for some \mathcal{H} . Similarly, $\mathcal{B} = \mathcal{B} *_N \mathcal{A}^{(1)} *_N \mathcal{A}$ if and only if $\mathcal{B} = \mathcal{G} *_N \mathcal{A}$ for some \mathcal{G} .*

An immediate consequence of the above result is shown next as a corollary.

Corollary 2.14. *Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$. Then*

- (i) $\mathcal{A} = \mathcal{A} *_N (\mathcal{A} *_N \mathcal{A})^{(1)} *_N (\mathcal{A} *_N \mathcal{A})$ and
- (ii) $\mathcal{A}^* = (\mathcal{A} *_N \mathcal{A}) *_N (\mathcal{A} *_N \mathcal{A})^{(1)} *_N \mathcal{A}^*$.

Another characterization of $\{1\}$ -inverse of a tensor is presented below.

Theorem 2.15. *If \mathcal{S} and \mathcal{T} are two invertible tensors, and \mathcal{G} is a $\{1\}$ -inverse of \mathcal{A} , then $\mathcal{T}^{-1} *_N \mathcal{G} *_N \mathcal{S}^{-1}$ is a $\{1\}$ -inverse of $\mathcal{B} = \mathcal{S} *_N \mathcal{A} *_N \mathcal{T}$. Moreover, every $\{1\}$ -inverse of \mathcal{B} is of this form.*

Proof. We have $\mathcal{T}^{-1} *_N \mathcal{G} *_N \mathcal{S}^{-1} \in \mathcal{B}\{1\}$ since $\mathcal{B} *_N (\mathcal{T}^{-1} *_N \mathcal{G} *_N \mathcal{S}^{-1}) *_N \mathcal{B} = \mathcal{S} *_N \mathcal{A} *_N \mathcal{T} *_N (\mathcal{T}^{-1} *_N \mathcal{G} *_N \mathcal{S}^{-1}) *_N \mathcal{S} *_N \mathcal{A} *_N \mathcal{T} = \mathcal{S} *_N \mathcal{A} *_N \mathcal{G} *_N \mathcal{A} *_N \mathcal{T} = \mathcal{S} *_N \mathcal{A} *_N \mathcal{T} = \mathcal{B}$. Again, let \mathcal{K} be any $\{1\}$ -inverse of \mathcal{B} . We also have $\mathcal{S}^{-1} *_N \mathcal{B} *_N \mathcal{T}^{-1} *_N (\mathcal{T} *_N \mathcal{K} *_N \mathcal{S}) *_N \mathcal{S}^{-1} *_N \mathcal{B} *_N \mathcal{T}^{-1} = \mathcal{S}^{-1} *_N \mathcal{B} *_N \mathcal{K} *_N \mathcal{B} *_N \mathcal{T}^{-1} = \mathcal{S}^{-1} *_N \mathcal{B} *_N \mathcal{T}^{-1}$. But $\mathcal{S}^{-1} *_N \mathcal{B} *_N \mathcal{T}^{-1} = \mathcal{A}$ as $\mathcal{B} = \mathcal{S} *_N \mathcal{A} *_N \mathcal{T}$. Considering $\mathcal{G} = \mathcal{S} *_N \mathcal{K} *_N \mathcal{T}$, we get $\mathcal{A} *_N \mathcal{G} *_N \mathcal{A} = \mathcal{A} *_N (\mathcal{S} *_N \mathcal{K} *_N \mathcal{T}) *_N \mathcal{A} = \mathcal{A} *_N \mathcal{S} *_N \mathcal{S}^{-1} *_N \mathcal{G} *_N \mathcal{T}^{-1} *_N \mathcal{A} = \mathcal{A} *_N \mathcal{G} *_N \mathcal{A} = \mathcal{A}$, i.e., $\{1\}$ -inverse of \mathcal{A} , and $\mathcal{K} = \mathcal{T}^{-1} *_N \mathcal{G} *_N \mathcal{S}^{-1}$. \square

Reverse order law for $\{1\}$ -inverse of tensors is shown next.

Theorem 2.16. *$(\mathcal{A} *_N \mathcal{B})^{(1)} = \mathcal{B}^{(1)} *_N \mathcal{A}^{(1)}$ if and only if*

$$(\mathcal{A}^{(1)} *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{B}^{(1)})^2 = \mathcal{A}^{(1)} *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{B}^{(1)}.$$

Proof. Suppose that $(\mathcal{A} *_N \mathcal{B})^{(1)} = \mathcal{B}^{(1)} *_N \mathcal{A}^{(1)}$. We then have

$$\mathcal{A} *_N \mathcal{B} *_N \mathcal{B}^{(1)} *_N \mathcal{A}^{(1)} *_N \mathcal{A} *_N \mathcal{B} = \mathcal{A} *_N \mathcal{B}.$$

Premultiplying and postmultiplying both sides by $\mathcal{A}^{(1)}$ and $\mathcal{B}^{(1)}$, respectively, we get

$$(\mathcal{A}^{(1)} *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{B}^{(1)})^2 = \mathcal{A}^{(1)} *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{B}^{(1)}.$$

□

We next have a corollary to the above result for rectangular matrices of suitable order.

Corollary 2.17. $(AB)^{(1)} = B^{(1)}A^{(1)}$ if and only if

$$(A^{(1)}ABB^{(1)})^2 = A^{(1)}ABB^{(1)}.$$

Let $\mathcal{A} = (a_{i_1 \dots i_N j_1 \dots j_M}) \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_M}$ and $\mathcal{B} = (b_{i_1 \dots i_N k_1 \dots k_M}) \in \mathbb{C}^{I_1 \times \dots \times I_N \times K_1 \times \dots \times K_M}$. Then *row block tensor* consisting of \mathcal{A} and \mathcal{B} is denoted by

$$[\mathcal{A} \ \mathcal{B}] \in \mathbb{C}^{\alpha^N \times \beta_1 \times \dots \times \beta_M}, \quad (2.1)$$

where $\alpha^N = I_1 \times \dots \times I_N$, $\beta_i = J_i + K_i$, $i = 1, \dots, M$, and is defined by

$$[\mathcal{A} \ \mathcal{B}]_{i_1 \dots i_N l_1 \dots l_M} = \begin{cases} a_{i_1 \dots i_N l_1 \dots l_M}, & i_1 \dots i_N \in [I_1] \times \dots \times [I_N], l_1 \dots l_M \in [J_1] \times \dots \times [J_M]; \\ b_{i_1 \dots i_N l_1 \dots l_M}, & i_1 \dots i_N \in [I_1] \times \dots \times [I_N], l_1 \dots l_M \in \Gamma_1 \times \dots \times \Gamma_M; \\ 0, & \text{otherwise.} \end{cases}$$

where $\Gamma_i = \{J_i + 1, \dots, J_i + K_i\}$, $i = 1, \dots, M$.

Let $\mathcal{C} = (c_{j_1 \dots j_M i_1 \dots i_N}) \in \mathbb{C}^{J_1 \times \dots \times J_M \times I_1 \times \dots \times I_N}$ and $\mathcal{D} = (d_{k_1 \dots k_M i_1 \dots i_N}) \in \mathbb{C}^{K_1 \times \dots \times K_M \times I_1 \times \dots \times I_N}$. Then *column block tensor* consisting of \mathcal{C} and \mathcal{D} is

$$\begin{bmatrix} \mathcal{C} \\ \mathcal{D} \end{bmatrix} = [\mathcal{C}^T \ \mathcal{D}^T]^T \in \mathbb{C}^{\beta_1 \times \dots \times \beta_M \times \alpha^N}. \quad (2.2)$$

For $\mathcal{A}_1 \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_M}$, $\mathcal{B}_1 \in \mathbb{C}^{I_1 \times \dots \times I_N \times K_1 \times \dots \times K_M}$, $\mathcal{A}_2 \in \mathbb{C}^{L_1 \times \dots \times L_N \times J_1 \times \dots \times J_M}$ and $\mathcal{B}_2 \in \mathbb{C}^{L_1 \times \dots \times L_N \times K_1 \times \dots \times K_M}$, we denote $\tau_1 = [\mathcal{A}_1 \ \mathcal{B}_1]$ and $\tau_2 = [\mathcal{A}_2 \ \mathcal{B}_2]$ as

the *row block tensors*. The *column block tensor* $\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$ can be written as

$$\begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{A}_2 & \mathcal{B}_2 \end{bmatrix} \in \mathbb{C}^{\rho_1 \times \dots \times \rho_N \times \beta_1 \times \dots \times \beta_M}, \quad (2.3)$$

where $\rho_i = I_i + L_i, i = 1, \dots, N; \beta_j = J_j + K_j$ and $j = 1, \dots, M$.

The product of some block tensors is recalled next from [18].

Lemma 2.18. (*Proposition 2.4, [18]*)

Let $[\mathcal{A} \ \mathcal{B}], \begin{bmatrix} \mathcal{C} \\ \mathcal{D} \end{bmatrix}$ and $\begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{A}_2 & \mathcal{B}_2 \end{bmatrix}$ be in the form as in equations (2.1), (2.2) and (2.3), respectively. Then

$$(a) \ \mathcal{F} *_N [\mathcal{A} \ \mathcal{B}] = [\mathcal{F} *_N \mathcal{A} \ \mathcal{F} *_N \mathcal{B}] \in \mathbb{C}^{\alpha^N \times \beta_1 \times \dots \times \beta_M};$$

$$(b) \ \begin{bmatrix} \mathcal{C} \\ \mathcal{D} \end{bmatrix} *_N \mathcal{F} = \begin{bmatrix} \mathcal{C} *_N \mathcal{F} \\ \mathcal{D} *_N \mathcal{F} \end{bmatrix} \in \mathbb{C}^{\beta_1 \times \dots \times \beta_M \times \alpha^N};$$

$$(c) \ [\mathcal{A} \ \mathcal{B}] *_M \begin{bmatrix} \mathcal{C} \\ \mathcal{D} \end{bmatrix} = \mathcal{A} *_M \mathcal{C} + \mathcal{B} *_M \mathcal{D} \in \mathbb{C}^{\alpha^N \times \alpha^N};$$

$$(d) \ \begin{bmatrix} \mathcal{C} \\ \mathcal{D} \end{bmatrix} *_N [\mathcal{A} \ \mathcal{B}] = \begin{bmatrix} \mathcal{C} *_N \mathcal{A} & \mathcal{C} *_N \mathcal{B} \\ \mathcal{D} *_N \mathcal{A} & \mathcal{D} *_N \mathcal{B} \end{bmatrix} \in \mathbb{C}^{\beta_1 \times \dots \times \beta_M \times \beta_1 \times \dots \times \beta_M};$$

$$(e) \ \begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{A}_2 & \mathcal{B}_2 \end{bmatrix} *_M \begin{bmatrix} \mathcal{C} \\ \mathcal{D} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_1 *_M \mathcal{C} & \mathcal{B}_1 *_M \mathcal{D} \\ \mathcal{A}_2 *_M \mathcal{C} & \mathcal{B}_2 *_M \mathcal{D} \end{bmatrix} \in \mathbb{C}^{\rho_1 \times \dots \times \rho_N \times \alpha^N};$$

$$(f) \ [\mathcal{G} \ \mathcal{H}] *_N \begin{bmatrix} \mathcal{A}_1 & \mathcal{B}_1 \\ \mathcal{A}_2 & \mathcal{B}_2 \end{bmatrix} = [\mathcal{G} *_N \mathcal{A}_1 + \mathcal{H} *_N \mathcal{A}_2 \ \mathcal{G} *_N \mathcal{B}_1 + \mathcal{H} *_N \mathcal{B}_2] \in \mathbb{C}^{S_1 \times \dots \times S_N \times \beta_1 \times \dots \times \beta_M};$$

where $\mathcal{F} \in \mathbb{C}^{\alpha^N \times \alpha^N}$, $\mathcal{G} \in \mathbb{C}^{S_1 \times \dots \times S_N \times I_1 \times \dots \times I_N}$ and $\mathcal{H} \in \mathbb{C}^{S_1 \times \dots \times S_N \times L_1 \times \dots \times L_N}$.

Our last result on $\{1\}$ -inverse of tensors is presented below.

Theorem 2.19. *For tensors \mathcal{A} and \mathcal{B} of suitable order, the following results hold.*

- (i) $[\mathcal{A} \ \mathcal{B}] *_N [\mathcal{A} \ \mathcal{B}]^{(1)} *_N \mathcal{A} = \mathcal{A}$.
- (ii) $(\mathcal{A} *_N \mathcal{A}^* + \mathcal{B} *_N \mathcal{B}^*) *_N (\mathcal{A} *_N \mathcal{A}^* + \mathcal{B} *_N \mathcal{B}^*)^{(1)} *_N \mathcal{A} = \mathcal{A}$, if $\mathcal{B}^{(1)} *_N \mathcal{A} = \mathcal{O}$ and $\mathcal{A}^{(1)} *_N \mathcal{B} = \mathcal{O}$.
- (iii) $(\mathcal{A} + \mathcal{U} *_N \mathcal{V})^{(1)} = \mathcal{A}^{(1)} - \mathcal{A}^{(1)} *_N \mathcal{U} *_N (\mathcal{I} + \mathcal{V} *_N \mathcal{A}^{(1)} *_N \mathcal{U})^{(1)} *_N \mathcal{V} *_N \mathcal{A}^{(1)}$ if $\mathcal{V} = \mathcal{V} *_N \mathcal{A}^{(1)} *_N \mathcal{A}$ and $\mathcal{U} = \mathcal{A} *_N \mathcal{A}^{(1)} *_N \mathcal{U}$.

$$\begin{aligned}
&= (\mathcal{A} + \mathcal{U} *_N \mathcal{V}) *_N \mathcal{A}^{(1)} *_N (\mathcal{A} + \mathcal{U} *_N \mathcal{V}) - (\mathcal{A} *_N \mathcal{A}^{(1)} *_N \mathcal{U} + \mathcal{U} *_N \mathcal{V} *_N \mathcal{A}^{(1)} *_N \mathcal{U}) *_N \\
&\quad (\mathcal{I} + \mathcal{V} *_N \mathcal{A}^{(1)} *_N \mathcal{U})^{(1)} *_N (\mathcal{V} *_N \mathcal{A}^{(1)} *_N \mathcal{A} + \mathcal{V} *_N \mathcal{A}^{(1)} *_N \mathcal{U} *_N \mathcal{V}) \\
&= \mathcal{A} + 2\mathcal{U} *_N \mathcal{V} + \mathcal{U} *_N \mathcal{V} *_N \mathcal{A}^{(1)} *_N \mathcal{U} *_N \mathcal{V} - \mathcal{V} *_N (\mathcal{I} + \mathcal{V} *_N \mathcal{A}^{(1)} *_N \mathcal{U}) *_N \mathcal{U} \\
&= \mathcal{A} + \mathcal{U} *_N \mathcal{V}.
\end{aligned}$$

□

The following corollary is obtained in case of rectangular matrices.

Corollary 2.20. *For matrices A and B of suitable order, the following results hold.*

- (i) $[A \ B][A \ B]^{(1)}A = A$.
- (ii) $(AA^* + BB^*)(AA^* + BB^*)^{(1)}A = A$ if $B^{(1)}A = O$ and $A^{(1)}B = O$.
- (iii) $(A + UV)^{(1)} = A^{(1)} - A^{(1)}U(I + VA^{(1)}U^{(1)})VA^{(1)}$ if $V = VA^{(1)}A$ and $U = AA^{(1)}U$.

Next result is for reflexive generalized inverse of a tensor $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$.

Lemma 2.21. *Let $\mathcal{Y}, \mathcal{Z} \in \mathcal{A}\{1\}$ and $\mathcal{X} = \mathcal{Y} *_N \mathcal{A} *_N \mathcal{Z}$. Then $\mathcal{X} \in \mathcal{A}\{1, 2\}$.*

Proof. To have this, we have to show that \mathcal{X} satisfies conditions (1) and (2) of Definition 2.1. So,

$$\begin{aligned}
\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} &= \mathcal{A} *_N (\mathcal{Y} *_N \mathcal{A} *_N \mathcal{Z}) *_N \mathcal{A} \\
&= (\mathcal{A} *_N \mathcal{Y} *_N \mathcal{A}) *_N \mathcal{Z} *_N \mathcal{A} \\
&= \mathcal{A} *_N \mathcal{Z} *_N \mathcal{A} = \mathcal{A}.
\end{aligned}$$

Again, we have

$$\begin{aligned}
\mathcal{X} *_N \mathcal{A} *_N \mathcal{X} &= (\mathcal{Y} *_N \mathcal{A} *_N \mathcal{Z}) *_N \mathcal{A} *_N (\mathcal{Y} *_N \mathcal{A} *_N \mathcal{Z}) \\
&= \mathcal{Y} *_N (\mathcal{A} *_N \mathcal{Z} *_N \mathcal{A}) *_N \mathcal{Y} *_N \mathcal{A} *_N \mathcal{Z} \\
&= \mathcal{Y} *_N (\mathcal{A} *_N \mathcal{Y} *_N \mathcal{A}) *_N \mathcal{Z} \\
&= \mathcal{Y} *_N \mathcal{A} *_N \mathcal{Z} = \mathcal{X}.
\end{aligned}$$

□

A characterization of class of $\{1, 3\}$ -inverse of $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$ is produced below.

Theorem 2.22. *The set $\mathcal{A}\{1, 3\}$ consists of all solutions \mathcal{X} of*

$$\mathcal{A} *_N \mathcal{X} = \mathcal{A} *_N \mathcal{A}^{(1,3)}.$$

Proof. Since $\mathcal{A}^{(1,3)}$ is a $\{1, 3\}$ -inverse of \mathcal{A} , so $\mathcal{A} *_N \mathcal{A}^{(1,3)} *_N \mathcal{A} = \mathcal{A}$ and $(\mathcal{A} *_N \mathcal{A}^{(1,3)})^* = \mathcal{A} *_N \mathcal{A}^{(1,3)}$. If $\mathcal{A} *_N \mathcal{X} = \mathcal{A} *_N \mathcal{A}^{(1,3)}$, then we have

$$\mathcal{A} *_N \mathcal{X} *_N \mathcal{A} = \mathcal{A} *_N \mathcal{A}^{(1,3)} *_N \mathcal{A} = \mathcal{A}$$

and

$$(\mathcal{A} *_N \mathcal{X})^* = (\mathcal{A} *_N \mathcal{A}^{(1,3)})^* = \mathcal{A} *_N \mathcal{A}^{(1,3)} = \mathcal{A} *_N \mathcal{X}.$$

So \mathcal{X} satisfies property (1) and (3) of Definition 2.1. Hence $\mathcal{X} \in \mathcal{A}\{1, 3\}$. On the other hand, suppose that $\mathcal{X} \in \mathcal{A}\{1, 3\}$. Then

$$\begin{aligned} \mathcal{A} *_N \mathcal{A}^{(1,3)} &= \mathcal{A} *_N \mathcal{X} *_N \mathcal{A} *_N \mathcal{A}^{(1,3)} \\ &= (\mathcal{A} *_N \mathcal{X})^* *_N \mathcal{A} *_N \mathcal{A}^{(1,3)} \\ &= \mathcal{X}^* *_N \mathcal{A}^* = \mathcal{A} *_N \mathcal{X}, \text{ as } (\mathcal{A}^{(1)})^* \in \mathcal{A}^*\{1\}. \end{aligned}$$

□

Similarly, we have the following one for $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$.

Theorem 2.23. *The set $\mathcal{A}\{1, 4\}$ consists of all solutions \mathcal{X} of*

$$\mathcal{X} *_N \mathcal{A} = \mathcal{A}^{(1,4)} *_N \mathcal{A}.$$

Here onwards, all our tensors in this section are assumed to be in $\mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$ unless otherwise mentioned. Equivalent conditions for a $\{1, 4\}$ -inverse is shown next.

Theorem 2.24. *The following three conditions are equivalent:*

- (i) $\mathcal{B} \in \mathcal{A}\{1, 4\}$.
- (ii) $\mathcal{B} *_N \mathcal{A} *_N \mathcal{A}^* = \mathcal{A}^*$.
- (iii) $\mathcal{B} *_N \mathcal{A} = \mathcal{A}^* *_N (\mathcal{A} *_N \mathcal{A}^*)^{(1)} *_N \mathcal{A}$.

Proof. (i) \Rightarrow (ii): From (i), we have $\mathcal{A} *_N \mathcal{B} *_N \mathcal{A} = \mathcal{A}$ and $(\mathcal{B} *_N \mathcal{A})^* = \mathcal{B} *_N \mathcal{A}$. Hence $(\mathcal{A} *_N \mathcal{B} *_N \mathcal{A})^* = \mathcal{A}^*$ yields $(\mathcal{B} *_N \mathcal{A})^* *_N \mathcal{A}^* = \mathcal{A}^*$ which implies $\mathcal{B} *_N \mathcal{A} *_N \mathcal{A}^* = \mathcal{A}^*$.

(ii) \Rightarrow (iii): Postmultiplying (ii) by $(\mathcal{A} *_N \mathcal{A}^*)^{(1)} *_N \mathcal{A}$, we get (iii).

(iii) \Rightarrow (i): Since $\mathcal{A}^* *_N (\mathcal{A} *_N \mathcal{A}^*)^{(1)} *_N \mathcal{A} *_N \mathcal{A}^* = \mathcal{A}^*$, so we obtain $\mathcal{A} *_N \mathcal{A}^* *_N (\mathcal{A} *_N \mathcal{A}^*)^{(1)} *_N \mathcal{A} = \mathcal{A}$ by taking conjugate transpose both sides. Hence $\mathcal{A} *_N \mathcal{B} *_N \mathcal{A} = \mathcal{A}$. Next $(\mathcal{B} *_N \mathcal{A})^* = (\mathcal{A}^* *_N (\mathcal{A} *_N \mathcal{A}^*)^{(1)} *_N \mathcal{A})^* = \mathcal{A}^* *_N (\mathcal{A} *_N \mathcal{A}^*)^{(1)} *_N \mathcal{A} = \mathcal{B} *_N \mathcal{A}$. \square

The matrix analogue is shown next for rectangular matrices of suitable order.

Corollary 2.25. *The following three conditions are equivalent:*

- (i) $B \in A\{1, 4\}$.
- (ii) $BAA^* = A^*$.
- (iii) $BA = A^*(AA^*)^{(1)}A$.

Sufficient conditions for reverse order law of $\{1, 4\}$ -inverse of tensors is presented next.

Theorem 2.26. *If $\mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{B}^*$ is hermitian, then*

$$(\mathcal{A} *_N \mathcal{B})^{(1,4)} = \mathcal{B}^{(1,4)} *_N \mathcal{A}^{(1,4)}.$$

Proof. The fact $\mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{B}^*$ is hermitian implies

$$\begin{aligned} \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{B}^* &= (\mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{B}^*)^* \\ &= (\mathcal{B} *_N \mathcal{B}^*)^* *_N (\mathcal{A}^{(1,4)} *_N \mathcal{A})^* \\ &= (\mathcal{B} *_N \mathcal{B}^*) *_N (\mathcal{A}^{(1,4)} *_N \mathcal{A}). \end{aligned}$$

Hence $\mathcal{A}^{(1,4)} *_N \mathcal{A}$ and $\mathcal{B} *_N \mathcal{B}^*$ are commutative. Let us consider $\mathcal{X} = \mathcal{B}^{(1,4)} *_N \mathcal{A}^{(1,4)}$ and $\mathcal{D} = \mathcal{A} *_N \mathcal{B}$. Then

$$\begin{aligned} \mathcal{D} *_N \mathcal{X} *_N \mathcal{D} &= \mathcal{A} *_N \mathcal{B} *_N \mathcal{B}^{(1,4)} *_N \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{B} \\ &= \mathcal{A} *_N \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{B}^{(1,4)} *_N \mathcal{B} \\ &= \mathcal{A} *_N \mathcal{B} = \mathcal{D}, \end{aligned}$$

since $\mathcal{A}^{(1,4)} \in \mathcal{A}\{1\}$ and $\mathcal{B}^{(1,4)} \in \mathcal{B}\{1\}$. Hence $\mathcal{X} \in \mathcal{D}\{1\}$, i.e., $\mathcal{B}^{(1,4)} *_N \mathcal{A}^{(1,4)} \in (\mathcal{A} *_N \mathcal{B})\{1\}$. Now

$$\begin{aligned}
& (\mathcal{B}^{(1,4)} *_N \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{B})^* \\
&= \mathcal{B}^* *_N (\mathcal{A}^{(1,4)} *_N \mathcal{A})^* *_N (\mathcal{B}^{(1,4)})^* \\
&= \mathcal{B}^{(1,4)} *_N \mathcal{B} *_N \mathcal{B}^* *_N \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N (\mathcal{B}^{(1,4)})^* \\
&= \mathcal{B}^{(1,4)} *_N \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{B}^* *_N (\mathcal{B}^{(1,4)})^*, \\
&= \mathcal{B}^{(1,4)} *_N \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{B} *_N (\mathcal{B}^{(1,4)} *_N \mathcal{B})^* \\
&= \mathcal{B}^{(1,4)} *_N \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{B}^{(1,4)} *_N \mathcal{B} \\
&= \mathcal{B}^{(1,4)} *_N \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{B}.
\end{aligned}$$

Thus $(\mathcal{A} *_N \mathcal{B})^{(1,4)} = \mathcal{B}^{(1,4)} *_N \mathcal{A}^{(1,4)}$. \square

The converse of the above theorem is not true, and is shown below with an example.

Example 2.27. Let $\mathcal{A} = (a_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$ and $\mathcal{B} = (b_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$ be two tensors such that

$$a_{ij11} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, a_{ij21} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, a_{ij12} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, a_{ij22} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix},$$

and

$$b_{ij11} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, b_{ij21} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b_{ij12} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, b_{ij22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

respectively. Then $\mathcal{A} *_N \mathcal{B} = (c_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, $\mathcal{A}^{(1,4)} = (x_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, and $\mathcal{B}^{(1,4)} = (y_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, where

$$c_{ij11} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}, c_{ij21} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, c_{ij12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, c_{ij22} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix},$$

$$x_{ij11} = \begin{pmatrix} -4 & 1 \\ -1 & 1 \end{pmatrix}, x_{ij21} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{pmatrix}, x_{ij12} = \begin{pmatrix} -\frac{1}{3} & -\frac{5}{6} \\ 0 & \frac{1}{6} \end{pmatrix}, x_{ij22} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

and

$$y_{ij11} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, y_{ij21} = \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & -\frac{5}{2} \end{pmatrix}, y_{ij12} = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, y_{ij22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

respectively. So $\mathcal{B}^{(1,4)} *_N \mathcal{A}^{(1,4)} = (d_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, where

$$d_{ij11} = \begin{pmatrix} -5 & -2 \\ \frac{1}{2} & \frac{15}{2} \end{pmatrix}, d_{ij21} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{6} & 0 \end{pmatrix}, d_{ij12} = \begin{pmatrix} -\frac{1}{3} & \frac{5}{12} \\ -\frac{5}{12} & \frac{1}{2} \end{pmatrix}, d_{ij22} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & -\frac{3}{2} \end{pmatrix}.$$

Hence

$$(\mathcal{A} *_N \mathcal{B})^{(1,4)} = \mathcal{B}^{(1,4)} *_N \mathcal{A}^{(1,4)}.$$

However, $\mathcal{A} *_N \mathcal{A}^{(1,4)} *_N \mathcal{B}^* *_N \mathcal{B}$ is not hermitian which can be seen from $\mathcal{A} *_N \mathcal{A}^{(1,4)} *_N \mathcal{B}^* *_N \mathcal{B} = (t_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, where

$$t_{ij11} = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}, t_{ij21} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, t_{ij12} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, t_{ij22} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and $(\mathcal{A} *_N \mathcal{A}^{(1,4)} *_N \mathcal{B}^* *_N \mathcal{B})^* = (\bar{t}_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, where

$$\bar{t}_{ij11} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \bar{t}_{ij21} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \bar{t}_{ij12} = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix}, \bar{t}_{ij22} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

As a corollary to Theorem 2.26, we obtain the following result for rectangular matrices of suitable order.

Corollary 2.28. *If $A^{(1,4)} A B B^*$ is hermitian, then*

$$(AB)^{(1,4)} = B^{(1,4)} A^{(1,4)}.$$

Similarly, one can have the following results for $\{1, 3\}$ -inverse of \mathcal{A} .

Theorem 2.29. *The following three conditions are equivalent:*

- (i) $\mathcal{B} \in \mathcal{A}\{1, 3\}$.
- (ii) $\mathcal{A}^* *_N \mathcal{A} *_N \mathcal{B} = \mathcal{A}^*$.
- (iii) $\mathcal{A} *_N \mathcal{B} = \mathcal{A} *_N (\mathcal{A}^* *_N \mathcal{A})^{(1)} *_N \mathcal{A}^*$.

Theorem 2.30. *If $\mathcal{A} *_N \mathcal{A}^{(1,3)} *_N \mathcal{B} *_N \mathcal{B}$ is hermitian, then*

$$(\mathcal{A} *_N \mathcal{B})^{(1,3)} = \mathcal{B}^{(1,3)} *_N \mathcal{A}^{(1,3)}.$$

The following example shows that the converse of the above result is not true.

Example 2.31. *Let $\mathcal{A} = (a_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$ and $\mathcal{B} = (b_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$ be two tensors such that*

$$a_{ij11} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, a_{ij21} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, a_{ij12} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, a_{ij22} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$b_{ij11} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, b_{ij21} = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, b_{ij12} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b_{ij22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

respectively. Then $\mathcal{A} *_N \mathcal{B} = (c_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, $\mathcal{A}^{(1,3)} = (x_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, and $\mathcal{B}^{(1,3)} = (y_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, where

$$c_{ij11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, c_{ij21} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, c_{ij12} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, c_{ij22} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$x_{ij11} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, x_{ij21} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, x_{ij12} = \begin{pmatrix} \frac{1}{2} & 0 \\ 1 & \frac{3}{2} \end{pmatrix}, x_{ij22} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

and

$$y_{ij11} = \begin{pmatrix} -2 & 1 \\ 2 & 2 \end{pmatrix}, y_{ij21} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, y_{ij12} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, y_{ij22} = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix},$$

respectively. So $\mathcal{B}^{(1,3)} *_N \mathcal{A}^{(1,3)} = (d_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, where

$$d_{ij11} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, d_{ij21} = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}, d_{ij12} = \begin{pmatrix} -\frac{5}{2} & \frac{1}{2} \\ \frac{7}{2} & 5 \end{pmatrix}, d_{ij22} = \begin{pmatrix} -2 & 0 \\ 3 & 4 \end{pmatrix}.$$

Hence

$$(\mathcal{A} *_N \mathcal{B})^{(1,3)} = \mathcal{B}^{(1,3)} *_N \mathcal{A}^{(1,3)},$$

But $\mathcal{A} *_N \mathcal{A}^{(1,3)} *_N \mathcal{B}^* *_N \mathcal{B}$ is not hermitian which can be seen from $\mathcal{A} *_N \mathcal{A}^{(1,3)} *_N \mathcal{B}^* *_N \mathcal{B} = (t_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, where

$$t_{ij11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, t_{ij21} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, t_{ij12} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, t_{ij22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

and $(\mathcal{A} *_N \mathcal{A}^{(1,3)} *_N \mathcal{B}^* *_N \mathcal{B})^* = (\bar{t}_{ijkl})_{1 \leq i,j,k,l \leq 2} \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$, where

$$\bar{t}_{ij11} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \bar{t}_{ij21} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \bar{t}_{ij12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \bar{t}_{ij22} = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$

We now present one of our important results, which yields a method of construction of the Moore-Penrose inverse of a tensor using $\{1, 3\}$ -inverse and $\{1, 4\}$ -inverse of \mathcal{A} . One can find the matrix version of these results in [1].

Theorem 2.32. *Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$. Then*

$$\mathcal{A}^\dagger = \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{A}^{(1,3)}.$$

Proof. Suppose that $\mathcal{X} = \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{A}^{(1,3)}$. Then

$$\begin{aligned} \mathcal{A} *_N \mathcal{X} *_N \mathcal{A} &= \mathcal{A} *_N (\mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{A}^{(1,3)}) *_N \mathcal{A} = \mathcal{A} *_N \mathcal{A}^{(1,3)} *_N \mathcal{A} = \mathcal{A}, \\ \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} &= (\mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{A}^{(1,3)}) *_N \mathcal{A} *_N (\mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{A}^{(1,3)}) \\ &= \mathcal{A}^{(1,4)} *_N (\mathcal{A} *_N \mathcal{A}^{(1,3)} *_N \mathcal{A}) *_N \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{A}^{(1,3)} \\ &= \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{A}^{(1,3)} \\ &= \mathcal{A}^{(1,4)} *_N (\mathcal{A} *_N \mathcal{A}^{(1,4)} *_N \mathcal{A}) *_N \mathcal{A}^{(1,3)} \\ &= \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{A}^{(1,3)} = \mathcal{X}, \\ (\mathcal{A} *_N \mathcal{X})^* &= (\mathcal{A} *_N (\mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{A}^{(1,3)}))^* \\ &= ((\mathcal{A} *_N \mathcal{A}^{(1,4)} *_N \mathcal{A}) *_N \mathcal{A}^{(1,3)})^* \\ &= (\mathcal{A} *_N \mathcal{A}^{(1,3)})^* \\ &= \mathcal{A} *_N \mathcal{A}^{(1,3)} \\ &= \mathcal{A} *_N \mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{A}^{(1,3)} \\ &= \mathcal{A} *_N \mathcal{X}, \end{aligned}$$

and

$$\begin{aligned}
(\mathcal{X} *_N \mathcal{A})^* &= ((\mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{A}^{(1,3)}) *_N \mathcal{A})^* \\
&= (\mathcal{A}^{(1,4)} *_N (\mathcal{A} *_N \mathcal{A}^{(1,3)} *_N \mathcal{A}))^* \\
&= (\mathcal{A}^{(1,4)} *_N \mathcal{A})^* \\
&= \mathcal{A}^{(1,4)} *_N (\mathcal{A} *_N \mathcal{A}^{(1,3)} *_N \mathcal{A}) \\
&= (\mathcal{A}^{(1,4)} *_N \mathcal{A} *_N \mathcal{A}^{(1,3)}) *_N \mathcal{A} = \mathcal{X} *_N \mathcal{A}.
\end{aligned}$$

Hence $\mathcal{X} = \mathcal{A}^\dagger$. \square

We remark that the first two conditions also follow from Lemma 2.21. The Moore-Penrose inverse of Kronecker product of two tensors can be computed using the next result.

Theorem 2.33. *Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$ and $\mathcal{B} \in \mathbb{C}^{K_1 \times \dots \times K_M \times L_1 \times \dots \times L_M}$. Then*

$$(\mathcal{A} \otimes \mathcal{B})^\dagger = \mathcal{A}^\dagger \otimes \mathcal{B}^\dagger.$$

Proof. Suppose that $\mathcal{K} = \mathcal{A} \otimes \mathcal{B}$ and $\mathcal{X} = \mathcal{A}^\dagger \otimes \mathcal{B}^\dagger$. We now have $\mathcal{K} *_N \mathcal{X} = (\mathcal{A} \otimes \mathcal{B}) *_N (\mathcal{A}^\dagger \otimes \mathcal{B}^\dagger) = \mathcal{A} *_N \mathcal{A}^\dagger \otimes \mathcal{B} *_N \mathcal{B}^\dagger$ by Lemma 1.4. So $\mathcal{K} *_N \mathcal{X} *_N \mathcal{K} = \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{B} *_N \mathcal{B}^\dagger *_N \mathcal{B} = \mathcal{A} \otimes \mathcal{B} = \mathcal{K}$, and $\mathcal{K} *_N \mathcal{X}$ is hermitian by Proposition 1.3 (a). Similarly, the other two conditions can be shown. \square

3. Multilinear system

Sylvester matrix equation plays significant roles in system and control theory [3, 12, 18]. One can compute exact solution of such an equation by using the Kronecker product, but the computational efforts rapidly increase with the dimensions of the matrices to be solved [3]. The Sylvester tensor equation via the Einstein product can be written in the following way:

$$\mathcal{A} *_N \mathcal{X} + \mathcal{X} *_M \mathcal{B} = \mathcal{D}, \quad (3.1)$$

where $\mathcal{A} \in \mathbb{R}^{I_1 \times \dots \times I_N \times I_1 \times \dots \times I_N}$, $\mathcal{X} \in \mathbb{R}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_M}$, $\mathcal{B} \in \mathbb{R}^{J_1 \times \dots \times J_M \times J_1 \times \dots \times J_M}$, and $\mathcal{D} \in \mathbb{R}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_M}$. This appears in the finite element method [6], finite difference or spectral method [9, 10], and plays an important role in

discretization of a linear partial differential equation in high dimension. Further, based on the operations of ‘block tensors’, one can write block tensor equation as:

$$\begin{bmatrix} \mathcal{A} & \mathcal{I}_1 \end{bmatrix} *_N \begin{bmatrix} \mathcal{X} & \mathcal{O} \\ \mathcal{O} & \mathcal{X} \end{bmatrix} *_N \begin{bmatrix} \mathcal{I}_2 \\ \mathcal{B} \end{bmatrix} = \mathcal{D}, \quad (3.2)$$

where $\mathcal{I}_1 = \mathbb{R}^{I_1 \times \dots \times I_N \times I_1 \times \dots \times I_N}$ and $\mathcal{I}_2 = \mathbb{R}^{J_1 \times \dots \times J_M \times J_1 \times \dots \times J_M}$ are unit tensors. Equivalently, we have

$$\mathcal{A} *_N \mathcal{X} *_M \mathcal{B} = \mathcal{D}, \quad (3.3)$$

where $\mathcal{A} = \mathbb{R}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$, $\mathcal{X} = \mathbb{R}^{J_1 \times \dots \times J_N \times K_1 \times \dots \times K_M}$, $\mathcal{B} = \mathbb{R}^{K_1 \times \dots \times K_M \times L_1 \times \dots \times L_M}$ and $\mathcal{D} = \mathbb{R}^{I_1 \times \dots \times I_N \times L_1 \times \dots \times L_M}$.

Sun *et al.*, [18] proved the following result for solving equation (3.3).

Theorem 3.1. (*Theorem 4.1, [18]*)

Let $\mathcal{A} \in \mathbb{R}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$, $\mathcal{X} \in \mathbb{R}^{J_1 \times \dots \times J_N \times K_1 \times \dots \times K_M}$, $\mathcal{B} \in \mathbb{R}^{K_1 \times \dots \times K_M \times L_1 \times \dots \times L_M}$ and $\mathcal{D} \in \mathbb{R}^{I_1 \times \dots \times I_N \times L_1 \times \dots \times L_M}$. Then the tensor equation $\mathcal{A} *_N \mathcal{X} *_M \mathcal{B} = \mathcal{D}$;

(a) is solvable if and only if there exist $\mathcal{A}^{(1)}$ and $\mathcal{B}^{(1)}$ such that

$$\mathcal{A} *_N \mathcal{A}^{(1)} *_N \mathcal{D} *_M \mathcal{B}^{(1)} *_M \mathcal{B} = \mathcal{D},$$

(b) in this case, the general solution is

$$\mathcal{X} = \mathcal{A}^{(1)} *_N \mathcal{D} *_M \mathcal{B}^{(1)} + \mathcal{Z} - \mathcal{A}^{(1)} *_N \mathcal{A} *_N \mathcal{Z} *_M \mathcal{B} *_M \mathcal{B}^{(1)}, \quad (3.4)$$

where $\mathcal{Z} \in \mathbb{R}^{J_1 \times \dots \times J_N \times K_1 \times \dots \times K_M}$ is an arbitrary tensor.

The next result is obtained as a corollary to the above one.

Corollary 3.2. Let $\mathcal{A} \in \mathbb{R}^{I_1 \times \dots \times I_M \times J_1 \times \dots \times J_N}$ and $\mathcal{A}^{(1)} \in \mathcal{A}\{1\}$. Then

$$\mathcal{A}\{1\} = \{\mathcal{A}^{(1)} + \mathcal{Y} - \mathcal{A}^{(1)} *_M \mathcal{A} *_N \mathcal{Y} *_M \mathcal{A} *_N \mathcal{A}^{(1)} : \mathcal{Y} \in \mathbb{R}^{J_1 \times \dots \times J_N \times I_1 \times \dots \times I_M}\}.$$

Proof. By Theorem 3.1, the general solution of $\mathcal{A} *_N \mathcal{X} *_M \mathcal{A} = \mathcal{A}$ is:

$$\mathcal{X} = \mathcal{A}^{(1)} *_M \mathcal{A} *_N \mathcal{A}^{(1)} + \mathcal{Z} - \mathcal{A}^{(1)} *_M \mathcal{A} *_N \mathcal{Z} *_M \mathcal{A} *_N \mathcal{A}^{(1)},$$

where \mathcal{Z} is arbitrary. Substituting $\mathcal{Z} = \mathcal{A}^{(1)} + \mathcal{Y}$, we get

$$\begin{aligned}
\mathcal{X} &= \mathcal{A}^{(1)} *_M \mathcal{A} *_N \mathcal{A}^{(1)} + \mathcal{A}^{(1)} + \mathcal{Y} - \mathcal{A}^{(1)} *_M \mathcal{A} *_N \mathcal{A}^{(1)} *_M \mathcal{A} *_N \mathcal{A}^{(1)} \\
&\quad - \mathcal{A}^{(1)} *_M \mathcal{A} *_N \mathcal{Y} *_M \mathcal{A} *_N \mathcal{A}^{(1)} \\
&= \mathcal{A}^{(1)} *_M \mathcal{A} *_N \mathcal{A}^{(1)} + \mathcal{A}^{(1)} + \mathcal{Y} - \mathcal{A}^{(1)} *_M \mathcal{A} *_N \mathcal{A}^{(1)} \\
&\quad - \mathcal{A}^{(1)} *_M \mathcal{A} *_N \mathcal{Y} *_M \mathcal{A} *_N \mathcal{A}^{(1)} \\
&= \mathcal{A}^{(1)} + \mathcal{Y} - \mathcal{A}^{(1)} *_M \mathcal{A} *_N \mathcal{Y} *_M \mathcal{A} *_N \mathcal{A}^{(1)}.
\end{aligned}$$

Hence

$$\mathcal{A}\{1\} = \{\mathcal{A}^{(1)} + \mathcal{Y} - \mathcal{A}^{(1)} *_M \mathcal{A} *_N \mathcal{Y} *_M \mathcal{A} *_N \mathcal{A}^{(1)} : \mathcal{Y} \in \mathbb{R}^{J_1 \times \dots \times J_N \times I_1 \times \dots \times I_M}\}.$$

□

The result produced hereunder is a special case of Theorem 3.1 in the setting of system of linear equations of tensors.

Corollary 3.3. *Let $\mathcal{A} \in \mathbb{R}^{I_1 \times \dots \times I_N \times J_1 \times \dots \times J_N}$ and $\mathcal{B} \in \mathbb{R}^{I_1 \times \dots \times I_N}$, then the equation*

$$\mathcal{A} *_N \mathcal{X} = \mathcal{B} \tag{3.5}$$

is consistent if and only if for some $\mathcal{A}^{(1)}$ such that

$$\mathcal{A} *_N \mathcal{A}^{(1)} *_N \mathcal{B} = \mathcal{B} \tag{3.6}$$

in which case the general solution of equation (3.5) is:

$$\mathcal{X} = \mathcal{A}^{(1)} *_N \mathcal{B} + (\mathcal{I} - \mathcal{A}^{(1)} *_N \mathcal{A}) *_N \mathcal{Y}, \tag{3.7}$$

for any arbitrary $\mathcal{Y} \in \mathbb{R}^{J_1 \times \dots \times J_N}$.

An alternative proof of Theorem 3.1 is provided below using Kronecker product of tensors.

Proof. The tensor equation $\mathcal{A} *_N \mathcal{X} *_M \mathcal{B} = \mathcal{D}$ can be rewritten as $(\mathcal{A} \otimes \mathcal{B}^*) *_{(N+M)} \mathcal{X} = \mathcal{D}$, where $\mathcal{A} \otimes \mathcal{B}^*$ is Kronecker product as defined in Section 2. Applying Corollary 3.3 to $(\mathcal{A} \otimes \mathcal{B}^*) *_{(N+M)} \mathcal{X} = \mathcal{D}$, we have the general solution of the form:

$$\mathcal{X} = (\mathcal{A} \otimes \mathcal{B}^*)^{(1)} *_{(N+M)} \mathcal{D} + [\mathcal{I} - (\mathcal{A} \otimes \mathcal{B}^*)^{(1)} *_M (\mathcal{A} \otimes \mathcal{B}^*)] *_{(N+M)} \mathcal{Z},$$

where \mathcal{Z} is arbitrary. Since we have $(\mathcal{A} \otimes \mathcal{B}^*)^{(1)} = \mathcal{A}^{(1)} \otimes (\mathcal{B}^*)^{(1)}$. Then, we get $(\mathcal{A} \otimes \mathcal{B}^*)^{(1)} *_M (\mathcal{A} \otimes \mathcal{B}^*) = (\mathcal{A}^{(1)} *_N \mathcal{A}) \otimes ((\mathcal{B}^*)^{(1)} *_M \mathcal{B}^*)$. Hence

$$\mathcal{X} = (\mathcal{A} \otimes \mathcal{B}^*)^{(1)} *_{(N+M)} \mathcal{D} + [\mathcal{I} - (\mathcal{A}^{(1)} *_N \mathcal{A}) \otimes ((\mathcal{B}^*)^{(1)} *_M \mathcal{B}^*)] *_{(N+M)} \mathcal{Z}.$$

Thus, we arrive at the general solution of the form

$$\mathcal{X} = \mathcal{A}^{(1)} *_N \mathcal{D} *_M \mathcal{B}^{(1)} + \mathcal{Z} - \mathcal{A}^{(1)} *_N \mathcal{A} *_N \mathcal{Z} *_M \mathcal{B} *_M \mathcal{B}^{(1)}.$$

□

The idea of the above proof is borrowed from the book [14] where the authors proved for matrices. Next two results are about solution of two and three tensor equations.

Theorem 3.4. *The tensor equations $\mathcal{A} *_N \mathcal{X} = \mathcal{B}$ and $\mathcal{X} *_N \mathcal{D} = \mathcal{F}$ has a common solution if and only if each equation separately has a solution and $\mathcal{A} *_N \mathcal{F} = \mathcal{B} *_N \mathcal{D}$.*

Proof. Let $\mathcal{X} = \mathcal{A}^{(1)} *_N \mathcal{B} + \mathcal{F} *_N \mathcal{D}^{(1)} - \mathcal{A}^{(1)} *_N \mathcal{A} *_N \mathcal{F} *_N \mathcal{D}^{(1)}$. Then $\mathcal{A} *_N \mathcal{X} = \mathcal{A} *_N \mathcal{A}^{(1)} *_N \mathcal{B}$ and $\mathcal{X} *_N \mathcal{D} = \mathcal{A}^{(1)} *_N \mathcal{B} *_N \mathcal{D} + \mathcal{F} *_N \mathcal{D}^{(1)} *_N \mathcal{D} - \mathcal{A}^{(1)} *_N \mathcal{A} *_N \mathcal{F} *_N \mathcal{D}^{(1)} *_N \mathcal{D}$. So \mathcal{X} is a common solution of both equations $\mathcal{A} *_N \mathcal{X} = \mathcal{B}$ and $\mathcal{X} *_N \mathcal{D} = \mathcal{F}$, provided $\mathcal{A} *_N \mathcal{F} = \mathcal{B} *_N \mathcal{D}$, $\mathcal{A} *_N \mathcal{A}^{(1)} *_N \mathcal{B} = \mathcal{B}$ and $\mathcal{F} *_N \mathcal{D}^{(1)} *_N \mathcal{D} = \mathcal{F}$. However, the last two tensor equations are equivalent to the consistency condition of Theorem 3.1 for the tensor equations $\mathcal{A} *_N \mathcal{X} = \mathcal{B}$ and $\mathcal{X} *_N \mathcal{D} = \mathcal{F}$. The other way is obvious. □

Theorem 3.5. *There is at most one tensor \mathcal{X} satisfying these three relations*

$$\mathcal{A} *_N \mathcal{X} = \mathcal{B}, \quad \mathcal{X} *_N \mathcal{A} = \mathcal{D} \text{ and } \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X}.$$

Proof. Suppose that there exists another tensor \mathcal{Y} satisfying these properties. Then

$$\begin{aligned} \mathcal{X} &= \mathcal{X} *_N \mathcal{A} *_N \mathcal{X} = \mathcal{X} *_N \mathcal{B} = \mathcal{X} *_N \mathcal{A} *_N \mathcal{Y} \\ &= \mathcal{D} *_N \mathcal{Y} = \mathcal{Y} *_N \mathcal{A} *_N \mathcal{Y} = \mathcal{Y}. \end{aligned}$$

□

Applying Theorem 3.1 to $\mathcal{A} *_N \mathcal{X} = \mathcal{A} *_N \mathcal{A}^{(1,3)}$ and putting $\mathcal{Z} = \mathcal{Y} + \mathcal{A}^{(1,3)}$, we have the following corollary which gives a characterization of class of $\{1, 3\}$ -inverse of \mathcal{A} .

Corollary 3.6. *Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_M \times J_1 \times \dots \times J_N}$ and $\mathcal{A}^{(1,3)} \in \mathcal{A}\{1, 3\}$. Then*

$$\mathcal{A}\{1, 3\} = \{\mathcal{A}^{(1,3)} + (\mathcal{I} - \mathcal{A}^{(1,3)} *_M \mathcal{A}) *_N \mathcal{Y} : \mathcal{Y} \in \mathbb{C}^{J_1 \times \dots \times J_N \times I_1 \times \dots \times I_M}\}.$$

Similar consideration of tensor system $\mathcal{X} *_M \mathcal{A} = \mathcal{A}^{(1,4)} *_M \mathcal{A}$ leads to the next corollary resulting a characterization of class of $\{1, 4\}$ -inverse of \mathcal{A} .

Corollary 3.7. *Let $\mathcal{A} \in \mathbb{C}^{I_1 \times \dots \times I_M \times J_1 \times \dots \times J_N}$ and $\mathcal{A}^{(1,4)} \in \mathcal{A}\{1, 4\}$. Then*

$$\mathcal{A}\{1, 4\} = \{\mathcal{A}^{(1,4)} + \mathcal{Y} *_M (\mathcal{I} - \mathcal{A} *_N \mathcal{A}^{(1,4)}) : \mathcal{Y} \in \mathbb{C}^{J_1 \times \dots \times J_N \times I_1 \times \dots \times I_M}\}. \quad (3.8)$$

We thus conclude this section with an analogous result of Corollary 3.3 where $\{1\}$ -inverse is replaced by the Moore-Penrose inverse.

Theorem 3.8. *$\mathcal{A} *_N \mathcal{X} = \mathcal{B}$ has a solution if and only if $\mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{B} = \mathcal{B}$. If a solution exists, then every solution is of the form*

$$\mathcal{X} = \mathcal{A}^\dagger *_N \mathcal{B} + (\mathcal{I} - \mathcal{A}^\dagger *_N \mathcal{A}) *_N \mathcal{W}, \quad (3.9)$$

where \mathcal{W} is arbitrary.

Proof. First, we verify the consistency condition. Clearly, if $\mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{B} = \mathcal{B}$ then $\mathcal{X}_0 = \mathcal{A}^\dagger *_N \mathcal{B}$ is a solution. Conversely, suppose that a solution exists. Then $\mathcal{A} *_N \mathcal{Y} = \mathcal{B}$ yields $\mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{Y} = \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{B}$ which implies $\mathcal{B} = \mathcal{A} *_N \mathcal{Y} = \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{B}$.

Now suppose that the system has a solution $\mathcal{X}_0 = \mathcal{A}^\dagger *_N \mathcal{B}$. Then, for $\mathcal{W} = \mathcal{X} - \mathcal{X}_0$, we have $\mathcal{A} *_N \mathcal{W} = \mathcal{A} *_N \mathcal{X} - \mathcal{A} *_N \mathcal{X}_0 = \mathcal{B} - \mathcal{A} *_N \mathcal{A}^\dagger *_N \mathcal{B} = \mathcal{B} - \mathcal{B} = \mathcal{O}$. But $\mathcal{A} *_N \mathcal{W} = \mathcal{O}$ implies $\mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{W} = \mathcal{O}$. Hence $\mathcal{W} = \mathcal{W} - \mathcal{A}^\dagger *_N \mathcal{A} *_N \mathcal{W} = (\mathcal{I} - \mathcal{A}^\dagger *_N \mathcal{A}) *_N \mathcal{W}$. Thus $\mathcal{X} = \mathcal{X}_0 + \mathcal{W} = \mathcal{A}^\dagger *_N \mathcal{B} + (\mathcal{I} - \mathcal{A}^\dagger *_N \mathcal{A}) *_N \mathcal{W}$. \square

4. Conclusion

We have further added some results on generalized inverses of tensors via the Einstein product to the existing theory. The matrix analogue of many

results presented in this paper are available in the famous book [1] and [14]. During the discussion, we encountered the following issues which have not been addressed in this paper, and are left as open problems for future studies. Looking at Remark 2 of Section 2, one may ask the question given next.

Question 1. When does $(\mathcal{A} *_N \mathcal{B})^\dagger = \mathcal{B}^\dagger *_N \mathcal{A}^\dagger$?

The above one can also be stated as reverse order law for Moore-Penrose inverse of tensors via the Einstein product. The other problem is noted below.

Question 2. Unlike matrices, does there exist a full rank factorization of tensors ? If so, can this be used to compute the Moore-Penrose inverse of a tensor \mathcal{A} ?

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